

Lecture 13. Light scattering and absorption by atmospheric particulate. Part 1: Main concepts: elementary wave, polarization, Stokes matrix, and scattering phase function. Rayleigh scattering.

Objectives:

1. Main concepts: elementary wave, polarization, Stoke matrix, and scattering phase function.
2. Rayleigh scattering.

Required Reading:

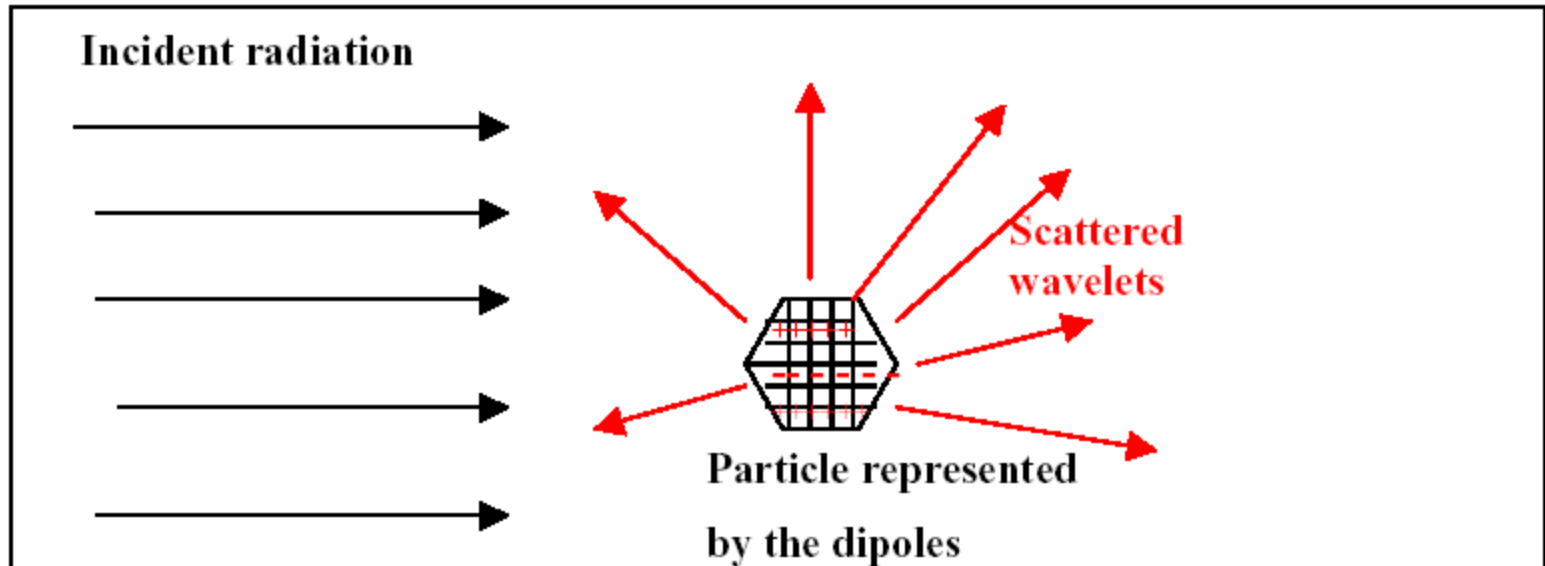
L02: 3.3.1, 5.3

Additional/advanced Reading:

Bohren, G.F., and D.R. Huffman, Absorption and scattering of light by small particles. John Wiley&Sons, 1983.

G. L. Stephens, Remote sensing of the lower atmosphere. Oxford Univ., 1994.

1. Main concepts: elementary wave and light beam, polarization, Stoke matrix, and scattering phase function.



Consider a single arbitrary particle. The incident electromagnetic field induces dipole oscillations. The dipoles oscillate at the frequency of the incident field and therefore scatter radiation in all directions. In a given direction of observation, the total scattered field is a superposition of the scattered wavelets of these dipoles. Scattering can be considered as two step process: (1) excitation and (2) reradiation.

- Scattering of the electromagnetic radiation is described by **the classical electromagnetic theory**, considering the propagation of a light beam as a transverse wave motion (collection of **electromagnetic individual waves**).
- Electromagnetic field is characterized by the **electric vector** \vec{E} and **magnetic vector** \vec{H} , which are orthogonal to each other and to the direction of the propagation. \vec{E} and \vec{H} obey the **Maxwell equations** (see Lecture 14).

Poynting vector gives the flow of radiant energy and the direction of propagation as (in cgs system)

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} \quad [13.1]$$

\vec{S} is in units of energy per unit time per unit area (i.e. flux);

NOTE: $\vec{E} \times \vec{H}$ means a **vector product** of two vectors.

Thus

$$I = \frac{1}{\Delta\Omega} \frac{c}{4\pi} |\vec{E}|^2$$

- Since electromagnetic field has wave-like nature, the classical theory of wave motion is used to characterize the propagation of radiation.

Consider a *plane wave* propagating in z-direction (i.e., E oscillates in the x-y plane).

The one-dimensional wave equation for a wave traveling with speed v is

$$\frac{\partial^2 \Psi}{\partial t^2} = v^2 \frac{\partial^2 \Psi}{\partial z^2}$$

The general solution is: $s(z, t) = f(vt - z) + g(vt + z)$ for *arbitrary* functions f and g .

One characteristic of the wave propagation is repetition of the E and H fields at set distance λ .

For a plane wave, assume displacements of electric field are simple harmonic. Then a wave traveling direction has a displacement

$$s_0(0, t_0) = s_0 \cos(\omega t_0 + \phi) \quad ,$$

where s_0 is the amplitude of the wave (energy carried by the wave is proportional to the square of this amplitude), $\omega = 2\pi v / \lambda$ is the angular frequency, and ϕ is the phase.

At a time $t = t_0 + \frac{z}{v}$ the disturbance has reached z and $s(z, t) = s(0, t_0)$, so

$$\begin{aligned} s(z, t) &= s_0 \cos \left[\omega \left(t - \frac{z}{v} \right) + \phi \right] = s_0 \cos \left(\omega t - \frac{z}{v} \omega + \phi \right) \\ &= s_0 \cos \left(\omega t - \frac{2\pi}{\lambda} z + \phi \right) = s_0 \cos(\omega t - kz + \phi) \end{aligned}$$

Using the Maclaurin's theorem:

$$\exp(\pm i\theta) = \cos(\theta) \pm i \sin(\theta)$$

the solution may be re-written in complex notation as

$$s(z, t) = s_0 \exp[-i(\omega t - kz + \phi)]$$

where the real part of this equation represents the wave.

The electric vector \vec{E} may be decomposed into the parallel E_l and perpendicular E_r components, so that

$$E_l = a_l \exp(-i\delta_l) \exp(-ikz + i\omega t) \quad [13.2a]$$

$$E_r = a_r \exp(-i\delta_r) \exp(-ikz + i\omega t) \quad [13.2b]$$

where a_l and a_r are the **amplitude** of the parallel E_l and perpendicular E_r components, respectively; δ_l and δ_r are the **phases** of the parallel E_l and perpendicular E_r components, respectively; k is the propagation (or wave) constant, $k = 2\pi/\lambda$, and ω is the circular frequency, $\omega = kc = 2\pi c/\lambda$

Eq.[13.2] can be written in cosine representation as

$$E_l = a_l \cos(\zeta + \delta_l)$$

$$E_r = a_r \cos(\zeta + \delta_r)$$

where $\zeta = kz - \omega t$ and $\zeta + \delta$ is called **phase**.

Then we have

$$E_l / a_l = \cos(\zeta) \cos(\delta_l) - \sin(\zeta) \sin(\delta_l) \quad [13.3]$$

$$E_r / a_r = \cos(\zeta) \cos(\delta_r) - \sin(\zeta) \sin(\delta_r)$$

and thus

$$(E_l / a_l)^2 + (E_r / a_r)^2 - 2(E_l / a_l)(E_r / a_r) \cos(\delta) = \sin^2(\delta) \quad [13.4]$$

where $\delta = \delta_r - \delta_l$ is the **phase difference** (or **phase shift**).

Eq.[13.4] represents an ellipse => **elliptically polarized wave**

If $\delta = m\pi$ ($m = 0, +1; +/-2, \dots$), then $\sin(\delta) = 0$ and Eq.[13.4] becomes

$$\left(\frac{E_l}{a_l} \pm \frac{E_r}{a_r} \right)^2 = 0 \quad \text{or} \quad \frac{E_l}{a_l} = \pm \frac{E_r}{a_r} \quad [13.5]$$

Eq.[13.5] represents two perpendicular lines => **linearly polarized wave**

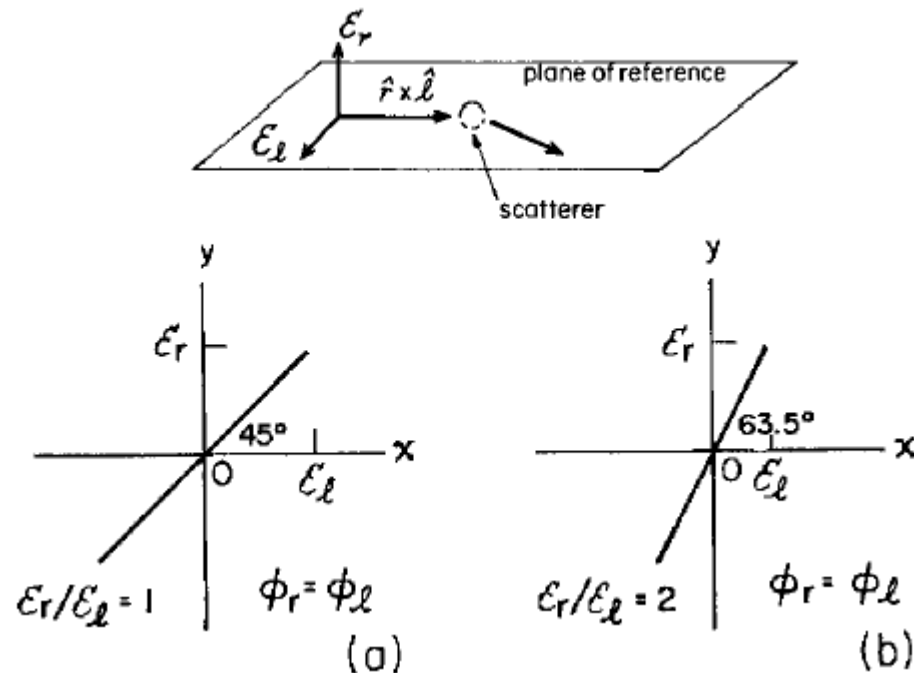
Illustration of the patterns the E field vector traces out. The E field has components in the parallel (ℓ) and perpendicular (r) directions. The type of polarization is determined by the relative amplitudes (\mathcal{E}) and phases (ϕ) of the two components. [Stephens, 1994; Fig. 2.9]

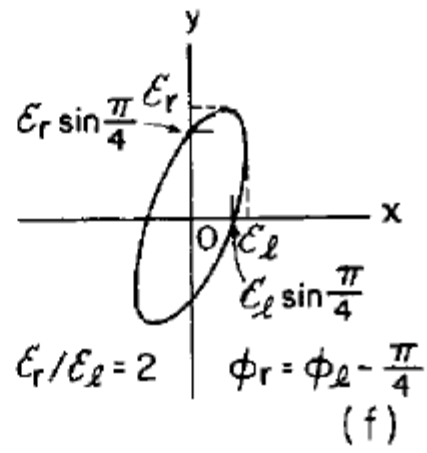
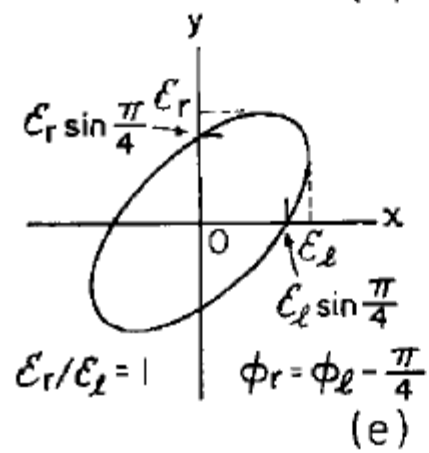
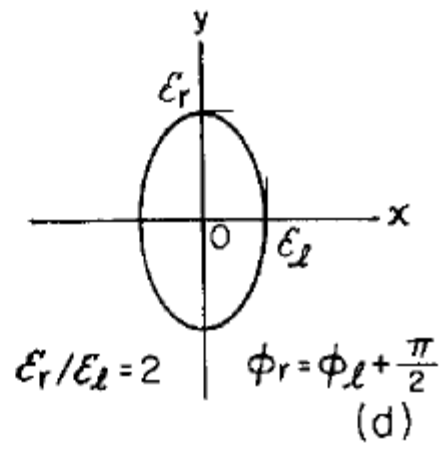
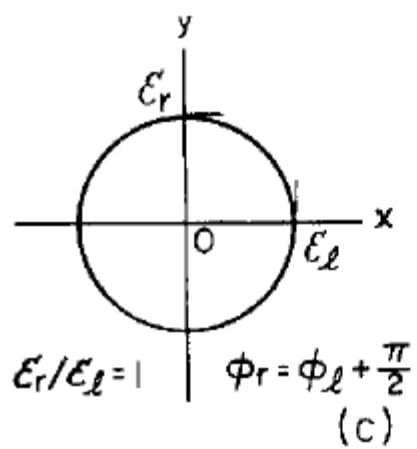
Tip of E field vector - E_{\perp} vs. E_{\parallel} :

linear polarization: $\Phi_{\parallel} = \Phi_{\perp}$

circular polarization: $\Phi_{\parallel} = \Phi_{\perp} + \pi/2$, $E_{0,\perp} = E_{0,\parallel}$

Single EM wave specified by E_{\parallel} , E_{\perp} , and $\Phi_{\parallel} - \Phi_{\perp}$.





- In general, light is a superposition of many waves of different frequencies, phases, and amplitudes. Polarization is determined by the relative size and correlations between two electrical field components. Radiation may be unpolarized, partially polarized, or completely polarized.
- **Natural light (sunlight) is unpolarized.**
- If there is a definite relation of phases between different scatterers \Rightarrow radiation is called **coherent**. If there is no relations in phase shift \Rightarrow light is called **incoherent**
- **Natural light (sunlight) is incoherent.**

The property of incoherent radiation:

The intensity due to all scattering centers is the sum of individual intensities.

NOTE: In our course, we study the **incoherent scattering of the atmospheric radiation**.

NOTE: The assumption of independent scatterers is violated if the particles are too closely packed (spacing between particles should be several times their diameters to prevent intermolecular forces from causing correlation between scattering centers).

- Eq.[13.4] shows that, in the general case, three independent parameters a_l , a_r and δ are required to characterize an electromagnetic wave. These parameters are not measured. Therefore, a new set of parameters (which are proportional to intensity) has been proposed by Stoke.

Stokes parameters

Stokes parameters: so-called intensity I , the degree of polarization Q , the plane of polarization U , and the ellipticity V of the electromagnetic wave

$$\begin{aligned}I &= E_l E_l^* + E_r E_r^* \\Q &= E_l E_l^* - E_r E_r^* \\U &= E_l E_r^* + E_r E_l^* \\V &= -i(E_l E_r^* - E_r E_l^*)\end{aligned}\tag{13.7}$$

They are related as

$$I^2 = Q^2 + U^2 + V^2\tag{13.8}$$

Stokes parameter can be also expressed as

$$\begin{aligned}I &= a_l^2 + a_r^2 \\Q &= a_l^2 - a_r^2 \\U &= 2 a_l a_r \cos(\delta) \\V &= 2 a_l a_r \sin(\delta)\end{aligned}\tag{13.9}$$

- Actual light consists of **many individual waves** each having its own amplitude and phase.

NOTE: During a second, a detector collects about millions of individual waves.

Measurable intensities are associated with the superposition of many millions of simple waves with independent phases. Therefore, for a light beam the Stokes parameters are averaged over a time period and may be represented as

$$\begin{aligned}
 I &= \langle a_l^2 \rangle + \langle a_r^2 \rangle = I_l + I_r \\
 Q &= \langle a_l^2 \rangle - \langle a_r^2 \rangle = I_l - I_r \\
 U &= \langle 2 a_l a_r \cos(\delta) \rangle \\
 V &= \langle 2 a_l a_r \sin(\delta) \rangle
 \end{aligned}
 \tag{13.10}$$

where $\langle \dots \rangle$ denote the time averaging.

For a light beam, we have

$$I^2 \geq Q^2 + U^2 + V^2 \tag{13.11}$$

The **degree of polarization** DP of a light beam is defined as

$$DP = (Q^2 + U^2 + V^2)^{1/2} / I \quad [13.12]$$

The **degree of linear polarization** LP of a light beam is defined by neglecting U and V as

$$LP = -\frac{Q}{I} = -\frac{I_l - I_r}{I_l + I_r} \quad [13.13]$$

Unpolarized light: $Q = U = V = 0$

Fully polarized light: $I^2 = Q^2 + U^2 + V^2$

Linear polarized light: $V = 0$

Circular polarized light: $|V| = I$

Elliptically polarized: $V \neq 0$.

Degree of polarization = $\sqrt{Q^2 + U^2 + V^2} / I$

Degree of linear polarization = $\sqrt{Q^2 + U^2} / I$

Degree of circular polarization = V / I

Stokes parameters have units of intensity; I is radiance.

Q and U require a reference plane, usually the scattering plane.

I is the sum of power in each component (I_{\parallel} and I_{\perp}).

Q is difference between the power in the two components.

U is correlation between the \parallel and \perp fields; max U for linear polarization at 45° .

V is correlation of \parallel with \perp shifted 90° in phase.

Vertical and horizontal polarization:

$$I_V = I_{\parallel} = (I + Q)/2 \quad I_H = I_{\perp} = (I - Q)/2$$

- The **scattering phase function $P(\cos\Theta)$** is defined as a non-dimensional parameter to describe the angular distribution of the scattered radiation as

$$\frac{1}{4\pi} \int_{\Omega} P(\cos \Theta) d\Omega = 1 \quad [13.14]$$

where Θ is called the **scattering angle** between the direction of incidence and observation.

NOTE: The phase function is expressed as

$$P(\cos\Theta) = P(\theta', \varphi', \theta, \varphi),$$

where (θ', φ') and (θ, φ) are the spherical coordinates of incident beam and direction of observation, and (see L02: Appendix C):

$$\cos(\Theta) = \cos(\theta')\cos(\theta) + \sin(\theta')\sin(\theta) \cos(\varphi' - \varphi) \quad [13.15]$$

Forward scattering refers to the observations directions for which $\Theta < \pi/2$

Backward scattering refers to the observations directions for which $\Theta > \pi/2$

2. Rayleigh scattering.

Consider a small homogeneous spherical particle (e.g., molecule) with size smaller than the wavelength of incident radiation \vec{E}_0 . Then the induced dipole moment \vec{p}_0 is

$$\vec{p}_0 = \alpha \vec{E}_0 \quad [13.16]$$

where α is the **polarizability** of the particle.

NOTE: Do not confuse the polarization of the medium with polarization associated with the EM wave.

According to the classical electromagnetic theory, the scattered electric field at the large distance r (called far field scattering) from the dipole is given (in cgs units) by

$$\vec{E} = \frac{1}{c^2} \frac{1}{r} \frac{\partial \vec{p}}{\partial t} \sin(\gamma) \quad [13.17]$$

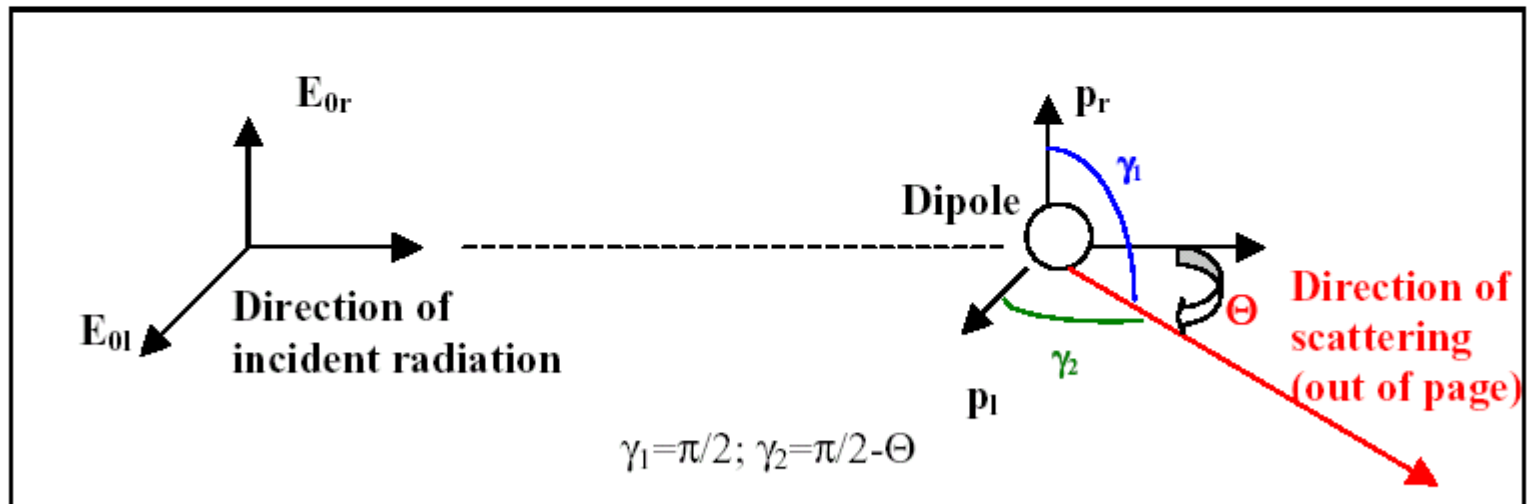
where γ is the angle between the scattered dipole moment \vec{p} and the direction of observation.

In an oscillating periodic field, the dipole moment is given in terms of induced dipole moment by

$$\vec{p} = \vec{p}_0 \exp(-ik (r - ct)) \quad [13.18]$$

and thus the electrical field is

$$\vec{E} = -\vec{E}_0 \frac{\exp(-ik (r - ct))}{r} k^2 \alpha \sin(\gamma) \quad [13.19]$$



NOTE: **Plane of scattering** (or **scattering plane**) is defined as a plane containing the incident beam and scattered beam in the direction of observation.

Decomposing the electrical vector on two orthogonal components perpendicular and parallel to the plane of scattering (a plane containing the incident and scattering beams), We have

$$E_r = -E_{0r} \frac{\exp(-ik(r - ct))}{r} k^2 \alpha \sin(\gamma_1) \quad [13.20]$$

$$E_l = -E_{0l} \frac{\exp(-ik(r - ct))}{r} k^2 \alpha \sin(\gamma_2)$$

Using that

$$I = \frac{1}{\Delta\Omega} \frac{c}{4\pi} |E|^2, \quad [13.21]$$

perpendicular and parallel intensities (or linear polarized intensities) are

$$I_r = I_{0r} k^4 \alpha^2 / r^2 \quad [13.22]$$

$$I_l = I_{0l} k^4 \alpha^2 \cos^2(\Theta) / r^2$$

Using that the natural light (incident beam) is not polarized ($I_{or}=I_{ol}=I_0/2$) and that $k=2\pi/\lambda$, we have

$$I = I_r + I_l = \frac{I_0}{r^2} \alpha^2 \left(\frac{2\pi}{\lambda} \right)^4 \frac{1 + \cos^2(\Theta)}{2} \quad [13.23]$$

Eq.[13.23] gives the intensity scattered by molecules for unpolarized incident light, Rayleigh scattering.

Rayleigh scattering phase function for incident unpolarized radiation (follows from Eq.[13.23]) is

$$P(\cos(\Theta)) = \frac{3}{4}(1 + \cos^2(\Theta)) \quad [13.24]$$

Eq.[13.23] may be rewritten in the form

$$I(\cos(\Theta)) = \frac{I_0}{r^2} \alpha^2 \frac{128 \pi^5}{3 \lambda^4} \frac{P(\Theta)}{4\pi} \quad [13.25]$$

Eq.[13.23] may be rewritten in the terms of the scattering cross section

$$I(\cos(\Theta)) = \frac{I_0}{r^2} \sigma_s \frac{P(\Theta)}{4\pi} \quad [13.26]$$

Here the scattering cross section (in units of area) by a single molecule is

$$\sigma_s = \alpha^2 \frac{128 \pi^5}{3 \lambda^4} \quad [13.27]$$

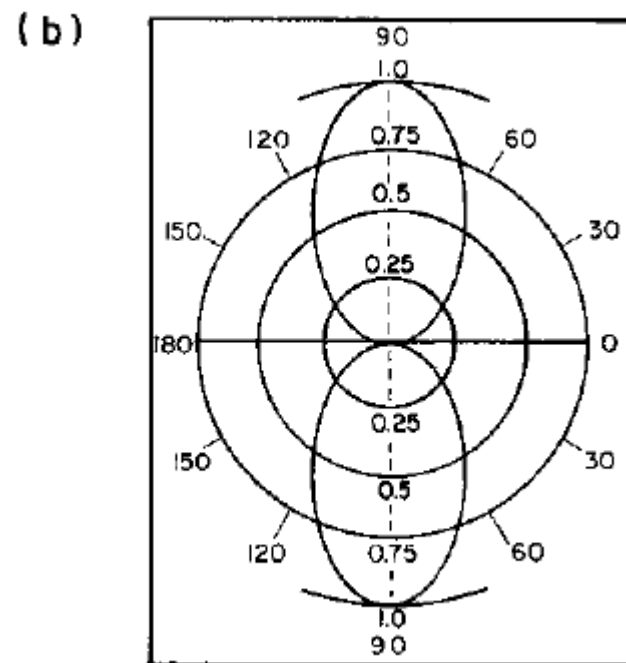
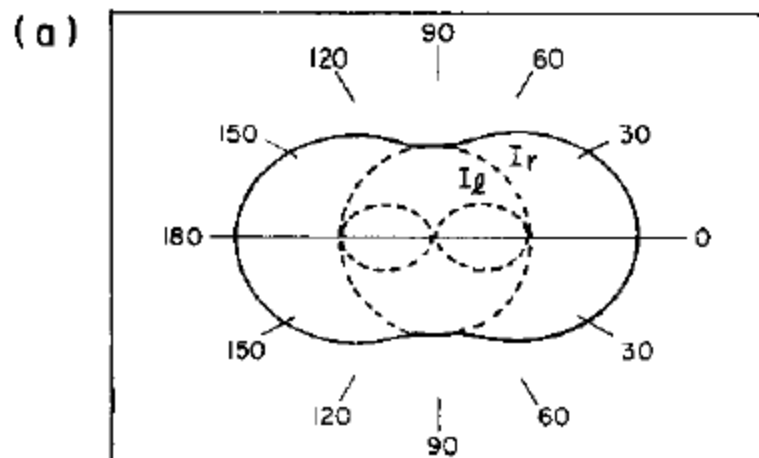
The polarizability α is given by the Lorentz-Lorenz formula (see L02: Appendix D):

$$\alpha = \frac{3}{4\pi N_s} \left(\frac{m^2 - 1}{m^2 + 2} \right) \quad [13.28]$$

where N is the number of molecules per unit volume and $m = m_r + im_i$ is the refractive index.

NOTE: For air molecules in solar spectrum m_r is about 1 but depends on λ , and $m_i = 0$ (see Lecture 14).

a) Polar diagram of the intensity of Rayleigh scattering for unpolarized incident radiation. b) Polar diagram of the degree of polarization of for Rayleigh scattering. [Stephens, 1994; Fig. 5.3]



Thus the polarizability can be approximated as

$$\alpha \approx \frac{1}{4\pi N_s} (m_r^2 - 1) \quad [13.29]$$

Therefore the scattering cross section of air molecules (Eq.[13.27]) becomes

$$\sigma_s = \frac{8\pi^3 (m_r^2 - 1)^2}{3\lambda^4 N_s^2} f(\delta) \quad [13.30]$$

where $f(\delta)$ is the correction factor for the anisotropic properties of air molecules, defined as $f(\delta) = (6+3\delta)/(6-7\delta)$ and $\delta=0.035$

Using this scattering cross section, one can calculate the optical depth of the entire atmosphere due to molecular scattering as

$$\tau(\lambda) = \sigma_s(\lambda) \int_0^{top} N(z) dz \quad [13.31]$$

Approximation of molecular Rayleigh optical depth (i.e., optical depth due to molecular scattering) down to pressure level p in the Earth's atmosphere:

$$\tau(\lambda) \approx 0.0088 \left(\frac{p}{1013 \text{ mb}} \right) \lambda^{-4.15+0.2\lambda} \quad [13.32]$$

Rayleigh scattering results in the sky polarization. The degree of linear polarization is

$$LP(\Theta) = -\frac{Q}{I} = -\frac{I_l - I_r}{I_l + I_r} = \frac{\cos^2 \Theta - 1}{\cos^2 \Theta + 1} = \frac{\sin^2 \Theta}{\cos^2 \Theta + 1} \quad [13.33]$$

Forward and backward scattering direction: unpolarized light

90° scattering angle: completely polarized

Complex Index of Refraction

Complex index of refraction, $m = n - i\kappa$, is the material property of a dielectric that determines its radiative properties.

Real part n is bulk refractivity - bends rays (Snells Law).

Imaginary part κ is bulk absorptivity: for bulk material, extinction is $\beta = \frac{4\pi\kappa}{\lambda}$.

Index of refraction depends on material and **wavelength**:

e.g. water in visible $m = 1.33$ (real)

water at $\lambda = 1.55 \text{ cm}$ (10°C) $m = 6.06 - 2.94i$.

Polarizability is complex if index m is complex:

→ dipole moment is then out of phase with applied field.

Charges oscillate out of phase with E field - "friction" in material.

Rayleigh Scattering from Spheres

Polarizability of sphere of radius r is

$$\alpha = \frac{m^2 - 1}{m^2 + 2} r^3$$

Rayleigh scattering efficiency factors for scattering and absorption:

$$Q_{sca} = \frac{8}{3} x^4 \left| \frac{m^2 - 1}{m^2 + 2} \right|^2 \quad Q_{abs} = -4x \operatorname{Im} \left\{ \frac{m^2 - 1}{m^2 + 2} \right\}$$

Rayleigh scattering cross sections:

$$C_{sca} = \frac{128\pi^5}{3} \frac{r^6}{\lambda^4} \left| \frac{m^2 - 1}{m^2 + 2} \right|^2 \quad C_{abs} = -\frac{8\pi^2 r^3}{\lambda} \operatorname{Im} \left\{ \frac{m^2 - 1}{m^2 + 2} \right\}$$

Scattering $\propto r^6$ (volume²). Absorption $\propto r^3$ (volume).

Rayleigh Scattering Example

Q_{abs} and Q_{sca} for $0.2 \mu\text{m}$ radius aerosols at $\lambda = 10 \mu\text{m}$.

Size parameter $x = \frac{2\pi r}{\lambda} = 2\pi(0.20)/10 = 0.126$.

	$(\text{NH}_4)_2\text{SO}_4$	NaCl
m	$2.190 - 0.130i$	$1.495 - 5.3 \times 10^{-8}i$
$\frac{m^2-1}{m^2+2}$	$0.56 - 0.037i$	$0.292 - 2.65 \times 10^{-8}i$
Q_{abs}	1.86×10^{-2}	1.33×10^{-8}
Q_{sca}	2.10×10^{-4}	5.66×10^{-5}

$$Q_{sca} = \frac{8}{3}(0.126)^4 |0.56 - 0.037i|^2 \quad Q_{abs} = -4(0.126) \text{Im} \{0.56 - 0.037i\}$$

In limit $x \rightarrow 0$ **absorption dominates scattering** if $|\text{Im}(m)| > 0$.